UNIVERSIDAD CARLOS III DE MADRID

working papers

Working Paper 10-37 (21)
Statistics and Econometrics Series
September 2010

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

RUIN PROBABILITIES IN A FINITE-HORIZON RISK MODEL WITH INVESTMENT AND REINSURANCE

Rosario Romera*† and Wolfgang Runggaldier‡**

Abstract

A finite horizon insurance model is studied where the risk/reserve process can be controlled by reinsurance and investment in the financial market. Obtaining explicit optimal solutions for the minimizing ruin probability problem is a difficult task. Therefore, we consider an alternative method commonly used in ruin theory, which consists in deriving inequalities that can be used to obtain upper bounds for the ruin probabilities and then choose the control to minimize the bound. We finally specialize our results to the particular, but relevant, case of exponentially distributed claims and compare for this case our bounds with the classical Lundberg bound

Keywords: Risk process, Reinsurance and investment, Lundberg's inequality.

* Rosario Romera (corresponding author), Departamento de Estadística, Universidad Carlos III de Madrid, C/ Madrid 126, 28903 Getafe (Madrid), e-mail: rosario.romera@uc3m.es

†Partially supported by the Spanish MEC grant SEJ2007-64500 and Comunidad de Madrid grant S2007/HUM0413

** Wolfgang Runggaldier, Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, e-mail: runggal@math.unipd.it

‡Part of the contribution by this author was obtained while he was visiting professor 2009 for the chair Quantitative Finance and Insurance at the LMU University in Munich funded by LMU Excellent.

Ruin probabilities in a finite-horizon risk model with investment and reinsurance

R. Romera and W. Runggaldier*
University Carlos III de Madrid and University of Padova

July 25, 2010

Abstract

A finite horizon insurance model is studied where the risk/reserve process can be controlled by reinsurance and investment in the financial market. Obtaining explicit optimal solutions for the minimizing ruin probability problem is a difficult task. Therefore, we consider an alternative method commonly used in ruin theory, which consists in deriving inequalities that can be used to obtain upper bounds for the ruin probabilities and then choose the control to minimize the bound. We finally specialize our results to the particular, but relevant, case of exponentially distributed claims and compare for this case our bounds with the classical Lundberg bound.

Keywords: Risk process, Reinsurance and investment, Lundberg's inequality.

MSC: 91B30, 93E20, 60J28.

*Partially supported by the Spanish MICINN grant# 123456. Part of the contribution by this author was obtained while he was visiting professor 2009 for the chair Quantitative Finance and Insurance at the LMU University in Munich funded by LMU Excellent. Hospitality and financial support are gratefully acknowledged.

1 Introduction

We consider an insurance risk/reserve process which can be controlled by reinsurance and investment in the financial market, and we study the ruin probability problem in the finite horizon case. Although controlling a risk/reserve process is a very active area of research (see Chen, Gerber and Shiu (2000), Wang, Yang and Wang (2004), Schmidli (2008), Huang, Zhao and Tang (2009) and references therein), obtaining explicit optimal solutions minimizing the ruin probability is a difficult task in a general setting even for the classical risk process. Thus, an alternative method commonly used in ruin theory is to derive inequalities for ruin probabilities. The inequalities can be used to obtain upper bounds for the ruin probabilities (see Willmot and Lin (2001), Grandell (2001), Schmidli (2002)), and this is the approach followed in the present paper.

Control problems for risk/reserve processes are commonly formulated in continuous time. Schäl (2004) introduces a formulation of the problem where events (arrivals of claims and asset price changes) occur at discrete points in time that may be deterministic or random, but their total number is fixed. Diasparra and Romera (2009) consider a similar formulation in discrete time. Having a fixed total number of events implies that in the case of random time points the horizon is random as well. Edoli and Runggaldier (2010) claim that a more natural way to formulate the problem in case of random time points is to consider a given fixed time horizon so that also the number of event times becomes random and this makes the problem nonstandard. Accordingly it is reasonable to assume that also the control decisions (level of reinsurance and amount invested) correspond to these random time points. Notice that this formulation can be seen equivalently in discrete or continuous time.

The rest of the paper is organized as follows. In Section 2 the risk model is formulated. In Section 3 some notation and basic definitions concerning the ruin probabilities are introduced. In Section 4 we present our main results on bounds for the ruin probability and on optimizing the bounds. Finally, in Section 5 we specialize our results to the case of exponentially distributed claims.

2 The model

We consider a finite time horizon $T > 0$. The stochastic elements that affect the evolution of the risk/reserve process are the timing and size of the claims as well as the evolution of the prices of the assets in which the insurer is investing. We allow for two possibilities to drive the evolution of the risk/reserve process: reinsurance and investment.

Claims occur at random points in time and also their sizes are random, while asset price evolutions are usually modeled as continuous time processes. On small time scales, prices actually change at discrete random time points and vary by tick size. In the proposed model we let also asset prices change only at discrete random time points with their sizes being random as well. This will allow us to consider the timing of the events to be triggered by a continuous-time semi-Markov process, i.e. a stochastic process where the embedded jump chain (the discrete process registering what values the process takes) is a Markov Chain, and where the holding times (time between jumps) are random variables, whose distribution function may depend on the two states between which the move is made. Since between event times the situation for the insurer does not change, we shall consider controls only at event times.

More precisely, to model the timing of the events (arrival of claims and asset price changes), inspired by Schäl (2004) we introduce the process $\{K_t\}_{t>0}$ for $t \leq T$, a continuous time semi-Markov process (SMP) on $\{0, 1\}$, where $K_t = 0$ holds for the *arrival of a claim*, and $K_t = 1$ for a *change in the asset price*. The embedded Markov chain, i.e.,the jump chain associated to the SMP $\{K_t\}_{t>0}$, evolves according to a transition probability matrix $P = \|p_{ij}\|_{i,j\in\{0,1\}}$ that is supposed to be given, and the holding times (time between jumps) are random variables whose probability distribution function may depend on the two states between which the move is made. We come back to this point in the next subsection 2.1.

Let T_n be the random time of the $n-th$ event, $n \geq 1$, and let the counting process N_t denote the number of events having occurred up to time t, defined as follows

$$N_t = \sum_{j=1}^{\infty} 1_{\{T_j \leq t\}}(1_{\{K_{T_j}=0\}} + 1_{\{K_{T_j}=1\}}) \tag{1}$$

and so

$$T_n = \inf\{t \geq 0 \,|\, N_t = n\}. \tag{2}$$

2.1 Risk process

In this section we introduce the dynamics of the controlled risk process X_t for $t \in [0,T]$ with T a given fixed horizon. For this purpose let Y_n be the $n-th\,(n \geq 1)$ claim payment represented by a sequence of independent and identically distributed (i.i.d.) random variables with common probability distribution function (p.d.f.) $F(y)$. Let Z_n be the random variable denoting the time between the occurrence of the $n-1$st and nth $(n \geq 1)$ jumps of the SMP $\{K_t\}_{t>0}$. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with p.d.f. $G(z)$. From this we may consider that the transition probabilities of the SMP $\{K_t\}_{t>0}$ are

$$P\{K_{T_{n+1}} = j, Z_{n+1} \leq s \mid K_{T_n} = i\} = p_{ij}G(s)$$

The risk process is controlled by reinsurance and investment. In general this means that we may choose adaptively at the event times T_{N_t} (they correspond to the jump times of N_t) the retention level (or proportionality factor or risk exposure) b_{N_t} of a reinsurance contract as well as the amount δ_{N_t} to be invested in the risky asset, namely in S_{N_t} with S_t denoting discounted prices. For the values b that the various b_{N_t} may take we assume that $b \in [b_{\min}, 1] \subset [0,1]$, where $b_{\min}$ will be introduced below and for the values of δ of the various δ_{N_t} we assume $\delta \in [\underline{\delta}, \bar{\delta}]$ with $\underline{\delta}$ and $\bar{\delta}$ exogenously given. Notice that this condition allows also for negative values of δ meaning that, see also Schäl (2004), short selling of stocks is allowed.

Assume that prices change only according to

$$\frac{S_{N_t+1} - S_{N_t}}{S_{N_t}} = \left(e^{W_{N_t+1}} - 1\right) K_{T_{N_t+1}}, \tag{3}$$

where W_n is an i.i.d. sequence taking values in $[\underline{w}, \bar{w}]$ with $\underline{w} < 0 < \bar{w}$ (it may also be all of $\mathbb{R}$) and with p.d.f. $H(w)$. For simplicity and without loss of generality we consider only one asset to invest in. An immediate generalization would be to allow for investment also in the money market account.

Let c be the premium rate (income) paid by the customer to the company, fixed in the contract. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b_{N_t} chosen at the various event times T_{N_t}, we denote by $C(b_{N_t})$ the net income rate of the insurer at time $t \in [0,T]$. For $b \in [b_{\min}, 1]$ we let $h(b, Y)$ represent the part of the generic claim Y paid by the insurer and in what follows we take the function $h(b, Y)$ to be of the form $h(b, Y) = b \cdot Y$ (proportional reinsurance). According to the expected value principle with safety loading θ of the reinsurer, the function $C(b)$ can be chosen as follows:

$$C(b) := c - (1+\theta)\frac{E\{Y_1 - h(b, Y_1)\}}{E\{Z_1 \wedge T\}}, \;\; 0 < t < T \tag{4}$$

We use Z_1 and Y_1 in the above formula since, by our independence assumption, the various Z_n and Y_n are all independent copies of Z_1 and Y_1. Notice also that, in order to keep formula (4) simple and possibly similar to standard usage, in the denominator of the right hand side we have considered the random time Z_1 between to successive events, while more correctly we should have taken the random time between two successive claims, which is larger. For this we can however play with the safety loading factor. In fact, if we denote by $\bar{Z}$ the average time between successive claims before T and, for a given θ put $\bar{\theta} = (1+\theta)\frac{\bar{Z}}{E\{Z_1 \wedge T\}} - 1$ we have that $\frac{(1+\theta)}{E\{Z_1 \wedge T\}} = \frac{(1+\bar{\theta})}{\bar{Z}}$. Since in this way $1 + \bar{\theta} = (1+\theta)\frac{\bar{Z}}{E\{Z_1 \wedge T\}}$ and $\bar{Z} > E\{Z_1 \wedge T\}$, we are assured that $(1+\bar{\theta}) > 1$. We can now define $b_{\min}$ as $b_{\min} := \min\{b \in [0,1] \mid c \geq C(b) \geq c^*\}$, where $c^* \geqslant 0$ denotes the minimal value of the premium considered by the insurer. We make the

Assumption 1 *Assume*

i) *The process* $\{K_{T_{N_t}}\}_{t \in [0,T]}$ *and the random variables* $(Z_n, Y_n, W_n)_{n \geq 1}$ *are all mutually independent.*

ii) $E\left\{e^{rY_1}\right\} < +\infty$ *for* $r \in (0, \bar{r})$ *with* $\bar{r} \in (0, \infty)$

iii) $c - (1+\theta)\frac{E\{Y_1\}}{E\{Z_1 \wedge T\}} \geq 0.$

Remark 2 *Notice that*

i) *Since* $b \leq 1$, *point ii) in Assumption 1 implies that also* $E\left\{e^{rbY_1}\right\} < +\infty$ *for* $r \in (0, \bar{r})$ *and all* $b \in [b_{min}, 1]$.

ii) *For* $h(b, Y) = bY$ *point iii) in Assumption 1 implies that* $c \geqslant C(b) \geqslant c^* \geqslant 0, \; \forall\, b \in [b_{min}, 1]$ *and that, furthermore,* $c \geqslant 0$.

Following the arguments in Remark 1 in Diasparra and Romera (2010) on the asymptotic optimality achieved by constant policies, in this paper we shall consider stationary constant policies $\pi = (b_{N_t}, \delta_{N_t}) \equiv (b, \delta)$, for $0 < t < T$, in which case $C(b_{N_t}) = C(b)$ for $0 < t < T$ and the optimization problem reduces then to a static optimization problem over the admissible values of (b, δ).

In the given setting we obtain now for the insurance risk process (surplus) X the following one-step transition dynamics between the generic random time T_n and T_{n+1} when at T_n a control action $\pi = (b, \delta)$ is taken for a certain $b \in [b_{\min}, 1] \subset [0, 1]$, and $\delta \in [\underline{\delta}, \bar{\delta}]$,

$$X_{T_{n+1}} = X_{T_n} + C(b)Z_{n+1} - (1 - K_{T_{n+1}})h(b, Y_{n+1}) + K_{T_{n+1}}\delta(e^{W_{n+1}} - 1) \quad (5)$$

We want now to express the one-step dynamics in (5) when starting from a generic time instant $t < T$ with a capital x. For this purpose note that if, for a given $t < T$ one has $N_t = n$, the time T_{N_t} is the random time of the $n - th$ event and $T_n \leq t \leq T_{n+1}$. Since, when standing at time t, we observe the time that has elapsed from the last event in T_{N_t}, it is not restrictive to assume that $t = T_{N_t}$ (see the comment below after (6)). Furthermore, since Z_n, Y_n, W_n are i.i.d., in the one-step random dynamics for the risk process X_t we may replace the generic $(Z_{n+1}, Y_{n+1}, W_{n+1})$ by (Z_1, Y_1, W_1). We may thus write

$$X_{N_t+1} = x + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1) \qquad (6)$$

for $0 < t < T$, $T > 0$ and with $X_t = x \geq 0$ (recall that we assumed $t = T_{N_t}$). Notice that, if we had $t \neq T_{N_t}$ and therefore $t > T_{N_t}$, the second term on the right in (6) would become $C(b)[Z_1 - (t - T_{N_t})]$ and (6) could then be rewritten as

$$X_{N_t+1} = [x - C(b)(t - T_{N_t})] + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1)$$

with the quantity $[x - C(b)(t - T_{N_t})]$, which is known at time t, replacing x. This is the sense in which above we mentioned that it is not restrictive to assume that $t = T_{N_t}$. In what follows we shall work with the risk process X_t, (or X_{N_t}) as defined by (6).

Following Schmidli (2008) we shall also introduce an absorbing (cemetery) state $\varkappa$, such that if $X_{N_t} < 0$ or $X_{N_t} = \varkappa$, then $X_{N_t+1} = \varkappa$, $\forall t \leq T$. The state space is denoted by $\mathfrak{X} = \mathbb{R} \cup \{\varkappa\}$.

3 Recursions

We start this section by specifying some notation and introducing the basic definitions concerning our ruin probabilities.

3.1 Notation and Definitions

Given $\pi = (b, \delta)$, we consider the following functions

$$
\begin{aligned}
u^{\pi}(y, z, w, k) \quad &: \; = (1-k)by - C(b)z - k\delta(e^{w} - 1) \\
\tau^{\pi}(y, w, k, x) \quad &: \; = \frac{(1-k)by - k(e^{w} - 1) - x}{C(b)}
\end{aligned}
\tag{7}
$$

so that $u^{\pi}(y, z, w, k) < x \iff z > \tau^{\pi}(y, w, k, x)$, as well as the disjoint sets

$$
\begin{aligned}
A^{+}_{x,\pi} &:= \{(y, z, w, k) | u^{\pi}(y, z, w, k) < x\} = \{(y, z, w, k) | \tau^{\pi}(y, w, k, x) < z\}, \\
A^{-}_{x,\pi} &:= \{(y, z, w, k) | u^{\pi}(y, z, w, k) \geq x\} = \{(y, z, w, k) | \tau^{\pi}(y, w, k, x) \geq z\}.
\end{aligned}
$$

The ruin probability over one intra-event period (namely the period between to successive event times) when using the stationary policy $\pi = (b, \delta)$ is, for a given initial surplus x at time $t \in (0, T)$ and initial event $K_{T_{N_t}} = k_0$,

$$
\psi^{\pi}_{1}(t, x; k_0) := \sum_{k_1=0}^{1} p_{k_0 k_1} \int_{\underline{w}}^{\bar{w}} \int_{0}^{\infty} G(\tau^{\pi}(y, w, k_1, x)) dF(y) dH(w). \tag{8}
$$

We want to obtain a recursive relation for

$$
\begin{aligned}
\psi^{\pi}_{n}(t, x; k_0) \quad &: \; = P^{\pi} \left\{ \bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\} \,|\, X_{N_t} = x, K_{T_{N_t}} = k_0 \right\} \\
&: \; = P^{\pi}_{x,k_0} \left\{ \bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\} \right\}
\end{aligned}
\tag{9}
$$

namely for the ruin probability when at most n events are considered in the interval $[t, T]$ and a stationary policy $\pi = (b, \delta)$ is adopted.

3.2 Recursive relations

In view of obtaining a recursive relation for $\psi_n^\pi(t,x,k_0)$, in addition to the sets $A^+_{x,\pi}$, $A^-_{x,\pi}$ define, for a given $t<T$, the events

$$B := \{X_{N_t+1} < 0\}\ ;\quad C := \bigcup_{h=N_t+2}^{(N_t+n)\wedge N_T} \{X_h < 0\} \tag{10}$$

and notice that $B\cap C=\emptyset$ and $C\cap\{N_T-N_t\le 1\}=\emptyset$. Furthermore, given (x,k), the event B is equivalent to an event happening in the set $A^-_{x,\pi}$.

The main result of this section is the recursive relation in the following

Proposition 3 *For an initial surplus x at a given time $t\in[0,T]$, as well as an initial event $K_{T_{N_t}}=k_0$ and a given stationary policy $\pi=(b,\delta)$, one has*

$$\begin{aligned}
&\psi_n^\pi(t,x,k_0)\\
&= P\{N_T-N_t>0\}\textstyle\sum_{k_1=0}^{1} p_{k_0k_1}\int_{\underline{w}}^{\bar{w}}\int_0^\infty G(\tau^\pi(y,w,k_1,x))dF(y)dH(w)+\\
&+P\{N_T-N_t>1\}\textstyle\sum_{k_1=0}^{1} p_{k_0k_1}\cdot\\
&\quad\cdot\textstyle\int_{\underline{w}}^{\bar{w}}\int_0^\infty\int_{\tau^\pi(y,w,k_1,x)}^{T-t}\psi_{n-1}^\pi(t+z,x-u^\pi(y,z,w,k_1),k_1)dG(z)dF(y)dH(w)
\end{aligned} \tag{11}$$

from which it immediately also follows that

$$\psi_1^\pi(t,x,k_0) = P\{N_T-N_t=1\}\sum_{k_1=0}^{1} p_{k_0k_1}\int_{\underline{w}}^{\bar{w}}\int_0^\infty G(\tau^\pi(y,w,k_1,x))dF(y)dH(w) \tag{12}$$

since in the case of at most one jump one has that $P\{N_T-N_t>0\} = P\{N_T-N_t=1\}$.

Proof: With the definitions in (10) we can write

$$\psi_n^\pi(t,x,k_0) = P^\pi_{x,k_0}\{B\cup C\} = P^\pi_{x,k_0}\{B\} + P^\pi_{x,k_0}\{C\} =$$

$$= P^\pi_{x,k_0}\{A^-_{x,\pi} \cap \{N_T - N_t > 0\}\} + P^\pi_{x,k_0}\{C \cap A^+_{x,\pi} \cap \{N_T - N_t > 1\}\} =$$

$$\begin{aligned}
&= P\{N_T - N_t > 0\} P^\pi_{x,k_0}\{\tau^\pi(Y_1, W_1, K_{T_{N_t+1}}, x) \geq Z_1 \text{ with } t + Z_1 \leq T\} + \\
&\quad + P\{N_T - N_t > 1\} \cdot \\
&\quad \cdot P^\pi_{x,k_0}\left\{\{(\tau^\pi, (Y_1, W_1, K_{T_{N_t+1}}, x) < Z_1 < T - t)\} \cap \left(\bigcup_{h=N_t+2}^{(N_t+n)\wedge N_T}\{X_h < 0\}\right)\right\} =
\end{aligned}$$

$$\begin{aligned}
&= P\{N_T - N_t > 0\} \sum_{k_1=0}^1 p_{k_0k_1} \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_0^{\tau^\pi(y,w,k_1,x)\wedge(T-t)} dG(z)dF(y)dH(w) + \\
&\quad + P\{N_T - N_t > 1\} \sum_{k_1=0}^1 p_{k_0k_1} \cdot \\
&\quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,k_1,x)}^{T-t} \psi^\pi_{n-1}(t+z, x - u^\pi(y,z,w,k_1), k_1) dG(z)dF(y)dH(w) =
\end{aligned}$$

$$\begin{aligned}
&= P\{N_T - N_t > 0\} \sum_{k_1=0}^1 p_{k_0k_1} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,k_1,x)) dF(y)dH(w) + \\
&\quad + P\{N_T - N_t > 1\} \sum_{k_1=0}^1 p_{k_0k_1} \cdot \\
&\quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,k_1,x)}^{T-t} \psi^\pi_{n-1}(t+z, x - u^\pi(y,z,w,k_1), k_1) dG(z)dF(y)dH(w)
\end{aligned}$$

∎

4 Bounds

We first derive bounds on the ruin probability in a general setting that we then specialize to particular distributions of the random variables that are involved as well as to the asymptotic case when $T \uparrow \infty$ that will be used for comparisons with the classical Lundberg bound.

4.1 General results

In this section we base ourselves on results in Diasparra and Romera (2009, 2010) that are here extended to the general setup of the present paper.

Given $\pi = (b, \delta)$ and defining (see (6)) for $t \in [0, T]$ the random variable

$$V_t^\pi := C(b) Z_1 \mathbf{1}_{\{Z_1 < T-t\}} - (1 - K_{T_{N_t+1}}) b Y_1 + K_{T_{N_t+1}} \delta \left(e^{W_1} - 1\right) \tag{13}$$

we make the following additional assumption

Assumption 4 *The distributions of Z, Y, W are such that there exists at least one stationary policy π for which*

$$\begin{array}{lll} i) & P\{V_t^\pi < 0\} > 0 & \text{for all} \quad t \in [0,T] \\ ii) & E\{V_t^\pi\} > 0 & \text{for all} \quad t \in [0,T] \end{array}$$

Notice that, if Y has a negative exponential distribution, point i) in Assumption 4 can easily be seen to be satisfied. We give now the following

Definition 5 *A stationary strategy $\pi = (b, \delta)$ will be called* admissible *if it satisfies Assumption 4. The set of stationary admissible strategies will be denoted by $\mathcal{A}$.*

For given $t \in [0,T]$, a given initial event $k_0 \in \{0,1\}$ at time $t = T_{N_t}$ and a given stationary policy $\pi = (b, \delta)$, let, for $r \in (0, \bar{r})$,

$$\ell_{k_0}^\pi(r) := E\left\{e^{-r\,V_t^\pi} \mid k_0\right\} - 1 \tag{14}$$

Proposition 6 *Under Assumption 4 we have*

i) *As a function of r, the $\ell_{k_0}^\pi(r)$ is convex with a negative slope at $r = 0$;*

ii) *the equation $\ell_{k_0}^\pi(r) = 0$ has a unique positive root $R^\pi(k_0)$ that we simply denote by R_0^π, so that the defining relation for R_0^π is*

$$\ell_{k_0}^\pi(R_0^\pi) = 0 \tag{15}$$

Proof: Differentiating under the expectation sign leads to

$$\begin{array}{l} \left(\ell_{k_0}^\pi\right)'(0) = \\ = E\left\{-\left[C(b)Z_1\mathbf{1}_{\{Z_1<T-t\}} - (1-K_{T_{N_t+1}})bY_1 + K_{T_{N_t+1}}\delta\left(e^{W_1}-1\right)\right] \mid k_0\right\} < 0 \\ \\ \left(\ell_{k_0}^\pi\right)''(r) = E\Big\{\left[C(b)Z_1\mathbf{1}_{\{Z_1<T-t\}} - (1-K_{T_{N_t+1}})bY_1 + K_{T_{N_t+1}}\delta\left(e^{W_1}-1\right)\right]^2 \cdot \\ \quad \cdot e^{-r\left[C(b)Z_1\mathbf{1}_{\{Z_1<T-t\}} - (1-K_{T_{N_t+1}})bY_1 + K_{T_{N_t+1}}\delta\left(e^{W_1}-1\right)\right]} \mid k_0\Big\} > 0 \end{array} \tag{16}$$

from which statement i) follows immediately. In view of ii) notice that (see also Lemma 4.1 in Schäl (2005) from point i) in Assumption 4 one obtains $\lim_{r\uparrow\bar{r}} \ell_{k_0}^\pi(r) = +\infty$. This fact, combined with i) leads to ii). ∎

Inspired by Diasparra and Romera (2009, 2010), define next

$$G^{\pi}(\theta) := \left(\frac{\int_0^{\theta} e^{-R_0^{\pi} C(b) z} dG(z)}{e^{-R_0^{\pi} C(b)\theta} G(\theta)} \right)^{-1} \cdot e^{R_0^{\pi} C(b)\theta} \int_0^{\theta} e^{-R_0^{\pi} C(b) z} dG(z) \tag{17}$$

Since, uniformly in π and θ,

$$\frac{\int_0^{\theta} e^{-R_0^{\pi} C(b) z} dG(z)}{e^{-R_0^{\pi} C(b)\theta} G(\theta)} \geq \frac{\int_0^{\theta} e^{-R_0^{\pi} C(b)\theta} dG(z)}{e^{-R_0^{\pi} C(b)\theta} G(\theta)} = \frac{e^{-R_0^{\pi} C(b)\theta} G(\theta)}{e^{-R_0^{\pi} C(b)\theta} G(\theta)} = 1$$

it follows that, uniformly in π,

$$G^{\pi}(\theta) \leq e^{R_0^{\pi} C(b)\theta} \int_0^{\theta} e^{-R_0^{\pi} C(b) z} dG(z) \leq e^{R_0^{\pi} C(b)\theta} E^{\pi} \left\{ e^{-R_0^{\pi} C(b) Z} \right\} \tag{18}$$

In view of the main result of this section, Theorem 10 below, we first prove

Lemma 7 *Given a surplus x at a given initial time $t \in [0, T]$ and an initial event k_0, we have*

$$\psi_1^{\pi}(t, x, k_0) \leq e^{-R_0^{\pi} x} \tag{19}$$

uniformly in π with R_0^{π} the unique positive root of (15).

Proof: Noticing that, whenever an event in $A_{x,\pi}^{-}$ occurs, then $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$, using (18) and the definition of $\tau^{\pi}(\cdot)$ in (7), from (12) we have

$$\begin{aligned}
&\psi_1^{\pi}(t, x, k_0) \\
&= P\{N_T - N_t = 1\} \textstyle\sum_{k_1=0}^{1} p_{k_0,k_1} \int_{\underline{w}}^{\bar{w}} \int_0^{\infty} G(\tau^{\pi}(y, w, k_1, x)) dF(y) dH(w) \\
&\leq P\{N_T - N_t = 1\} \textstyle\sum_{k_1=0}^{1} p_{k_0,k_1} E^{\pi} \left\{ e^{-R_0^{\pi} C(b) Z} \right\} \\
&\cdot \int_{\underline{w}}^{\bar{w}} \int_0^{\infty} \exp\left[\frac{R_0^{\pi} C(b)}{C(b)} \left((1 - k_1) b y - k_1 \delta \left(e^{w} - 1 \right) - x \right) \right] dF(y) dH(w) \\
&= P\{N_T - N_t = 1\} \\
&\quad e^{-R_0^{\pi} x} E^{\pi} \left\{ e^{-R_0^{\pi} C(b) Z_1} \right\} E \left\{ e^{R_0^{\pi} \left[(1 - K_{T_{N_t+1}}) b Y_1 - K_{T_{N_t+1}} \delta \left(e^{W_1} - 1 \right) \right]} \mid k_0 \right\} \\
&\leq P\{N_T - N_t = 1\} e^{-R_0^{\pi} x}
\end{aligned}$$

where in the next-to-last relation we have used the fact that $\ell_{k_0}^{\pi}(R_0^{\pi}) = 0$. ∎

Lemma 8 *For given* (t,x) *we have*

$$\psi_n^\pi(t,x,k_0) \le \gamma_n e^{-R^\pi x} \tag{20}$$

uniformly in $n \in \mathbb{N}, \pi,$ *and* $k_0 \in \{0,1\}$, *where*

$$R^\pi := \min\{R_0^\pi, R_1^\pi\} \tag{21}$$

with R_k^π, $(k=0,1)$ *the unique positive solution of* $\ell_k^\pi(R_k^\pi) = 0$ *(see (15)) and* γ_n *defined recursively by*

$$\begin{cases} \gamma_1 = 1 \\ \gamma_n = \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\} \end{cases} \tag{22}$$

Remark 9 *Due to the defining relations (22) it follows immediately that* $\gamma_n \le 1$ *for all* $n \in \mathbb{N}$. *In fact, using forward induction, we see that the inequality is true for* $n = 1$ *and, assuming it true for* $n-1$, *we have*

$$\gamma_n = \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\} \le P\{N_T - N_t > 0\} \le 1 \tag{23}$$

Proof: Before proving (20) let us recall that, for given t, a given $k_0 = K_{T_{N_t}} \in \{0,1\}$ and with V_t^π as in (13),

$$E\left\{e^{-r V_t^\pi} \mid k_0\right\} = \ell_{k_0}^\pi(r) + 1 \tag{24}$$

On the other hand, $\ell_{k_0}^\pi(r)$ has (see Proposition 6) a negative slope in $r=0$ and is convex so that, being $\ell_{k_0}^\pi(R_0^\pi) = 0$, for $r \in [0, R_0^\pi]$ one has $\ell_{k_0}^\pi(r) < 0$. Given that $R^\pi := \min[R_0^\pi, R_1^\pi]$, it follows that

$$\begin{cases} \text{if} \quad R^\pi = R_0^\pi \ \text{ then } \ \ell_{k_0}^\pi(R_0^\pi) + 1 = 1 \\ \\ \text{if} \quad R^\pi = R_1^\pi < R_0^\pi \ \text{ then } \ \ell_{k_0}^\pi(R^\pi) + 1 < 1 \end{cases}$$

Concluding we have that

$$E\left\{e^{-R^\pi V_t^\pi} \mid k_0\right\} \le 1 \tag{25}$$

The proof of (20) now proceeds by induction. By Lemma 7 the statement is true for $n=1$ and notice that, by the definition of R^π, $e^{-R_0^\pi x} \le e^{-R^\pi x}$. Assume it holds true for $n-1$. From Proposition 3, whereby for the first term on the right hand side of (11) one uses Lemma 7 (see also its proof), noticing also that on the event $z \le \tau^\pi(y,w,k,x)$ one has $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$ and that $e^{-R_0^\pi x} \le e^{-R^\pi x}$, it follows that

$$
\begin{aligned}
&\psi_n^\pi(t,x,k_0) = \\
&= P\{N_T - N_t = 1\} e^{-R_0^\pi x} + \\
&\quad + P\{N_T - N_t > 1\} \textstyle\sum_{k_1=0}^{1} p_{k_0,k_1} \cdot \\
&\quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,k_1,x)}^{T-t} \psi_{n-1}^\pi(t+z, x-u^\pi(y,z,w,k_1), k_1) dG(z) dF(y) dH(w) \\
&\le P\{N_T - N_t = 1\} e^{-R^\pi x} \\
&\quad + P\{N_T - N_t > 1\} \gamma_{n-1} \\
&\quad \textstyle\sum_{k_1=0}^{1} p_{k_0,k_1} \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_0^{T-t} e^{-R^\pi [x-u^\pi(y,z,w,k_1)]} dG(z) dF(y) dH(w) \\
&\le P\{N_T - N_t = 1\} e^{-R^\pi x} \\
&\quad + P\{N_T - N_t > 1\} \gamma_{n-1} \\
&\quad e^{-R^\pi x} E\left\{ e^{-R^\pi \left[C(b) Z_1 \mathbf{1}_{\{Z_1 < T-t\}} - (1-K_{T_{N_t+1}}) b Y_1 + K_{T_{N_t+1}} \delta\left(e^{W_1}-1\right)\right]} \mid k_0 \right\} \\
&\le (P\{N_T - N_t = 1\} + P\{N_T - N_t > 1\} \gamma_{n-1}) e^{-R^\pi x} \le e^{-R^\pi x}
\end{aligned}
$$

where in the next-to-last two expression we have used (25). ∎

We come now to our main result in this section, namely Theorem 10 whose proof follows immediately from Lemma 8 noticing that, see Remark 9, one has $\gamma_n \le 1$.

Theorem 10 *Given an initial surplus x at a given time $t \in [0,T]$, we have, for all $n \in \mathbb{N}$ and any initial event $k_0 \in \{0,1\}$ and uniformly in the control π*

$$\psi_n^\pi(t,x,k_0) \le e^{-R^\pi x}$$

where $R^\pi := \min[R_0^\pi, R_1^\pi]$.

4.2 Optimizing the bounds

Given the one-step random dynamics for the risk process X_t as specified in (6) for given $t = T_{N_t} < T$ and admissible control action $\pi = (b,\delta)$, consider the random variable (see (13))

$$V_t^\pi := C(b) Z_1 \mathbf{1}_{\{Z_1 < T-t\}} - (1 - K_{T_{N_t+1}}) b Y_1 + K_{T_{N_t+1}} \delta(e^{W_1} - 1) \qquad (26)$$

The arguments in Remark 1 in Diasparra and Romera (2010) that, see Section 2.1, led us consider stationary constant policies $\pi = (b_{N_t}, \delta_{N_t}) \equiv (b, \delta)$, for $0 < t < T$, are based on a stationarity property of the underlying processes that motivate us to make the following additional

Assumption 11 *For $T \uparrow \infty$ the semi-Markov process $\{K_T\}$ achieves a stationary regime. More specifically, there exists Q_0 such that*

$$Pr\{K_\infty = 0\} = Q_0 \ ; \quad Pr\{K_\infty = 1\} = Q_1 := 1 - Q_0$$

It is intuitively clear that the ruin probability, and consequently also the bound on the ruin probability, will decrease if one increases the quantity in (26). Given the stationarity assumption for the strategy of reinsurance and investment, as well as the i.i.d. property of the sequences Z_n, Y_n, W_n and Assumption 11, our static optimization problem becomes the following

Problem: Determine a stationary admissible strategy strategy $\pi = (b, \delta)$ that, for a given initial time t maximizes $E\{V_t^\pi\}$ i.e.

$$\textbf{Determine } argmax_{\pi \in \mathcal{A}} E\{V_t^\pi\}$$

Remark 12 *The remark concerns the following:*

R.1 *Notice that, from the stationarity of the strategy, the proportional reinsurance assumption, and from the i.i.d. property of Z_n and Y_n, from (4) we have*

$$C(b) := c - (1+\theta)\frac{(1-b)E\{Y_1\}}{E\{Z_1 \wedge T\}}$$

From (26) and Assumption 11 we then obtain

$$\begin{array}{rcl} E\{V_t^\pi\} & = & c - (1+\theta)\frac{(1-b)E\{Y_1\}}{E\{Z_1 \wedge T\}} \cdot E\{Z_1 \wedge T\} \\ & - & Q_0 b E\{Y_1\} + Q_1 \delta \left(E\{e^{W_1}\} - 1\right) \\ & = & c - (1+\theta)E\{Y_1\} \\ & + & b\, E\{Y_1\}\, [(1+\theta) - Q_0] \\ & + & \delta\, Q_1 \left(E\{e^{W_1}\} - 1\right) \end{array} \tag{27}$$

R.2 *Recall from (16) in the proof of Proposition 6 that for the derivative $\dot{\ell}_{k_0}^\pi(0)$ at $r = 0$ we have*

$$\dot{\ell}_{k_0}^\pi(0) = E\{-V_t^\pi\} < 0$$

where the negativity follows from Assumption 4 ii) for an admissible stationary $\pi \in \mathcal{A}$. Notice also that, if $\mathcal{A}$ is nonempty, by maximizing $E\{V_t^\pi\}$, we always obtain a negative value for $\dot{\ell}_{k_0}^\pi(0)$. Recall then from Proposition 6 i) that $\ell_{k_0}^\pi(r)$ is convex with a negative slope at $r = 0$. By maximizing $E\{V_t^\pi\}$ we achieve thus the steepest negative slope of $\ell_{k_0}^\pi(r)$ at $r = 0$ which would imply the largest possible value R_0^π for the solution of (15) and thus the sharpest bound in Theorem 10, provided that by maximizing $E\{V_t^\pi\}$ one would not increase the convexity of $\ell_{k_0}^\pi(r)$. To this effect notice that from the second relation in (16) we obtain that

$$\ddot{\ell}_{k_0}^\pi(r) = E[(V_t^\pi)^2 e^{-rV_t^\pi} | k_0]$$

which expresses the sharpness of the curvature of the convex function $\ell_{k_0}^\pi(r)$. It is now easily seen that for large values of V_t^π, which will result from maximizing $E\{V_t^\pi\}$, this curvature is small so that indeed, by maximizing $E\{V_t^\pi\}$, one gets the largest possible value R_0^π for the solution of (15) and thus the sharpest bound in Theorem 10.

From the rightmost expression in (27), which is linear in δ and, due to the proportional reinsurance assumption, linear also in b, we get that the optimal stationary strategy, maximizing $E\{V_t^\pi\}$ and thus the bound on the ruin probability in Theorem 10, takes on extreme values. Since furthermore $Q_0 < 1 + \theta$, we immediately obtain the following

Proposition 13 *In the context and under the assumptions of the paper, the optimal stationary strategy that minimizes the bound on the ruin probability in Theorem 10 is given by*

$$b^* = b_{max} = 1 \quad \textit{i.e. no reinsurance}$$
$$\delta^* = \begin{cases} \underline{\delta} & \textit{if} \quad E\{e^{W_1}\} < 1 \ \textit{i.e. if prices tend to decrease} \\ \bar{\delta} & \textit{if} \quad E\{e^{W_1}\} > 1 \ \textit{i.e. if prices tend to increase} \end{cases} \tag{28}$$

Remark 14 *Notice that*

R.1 *The extreme reinvestment decision $b^* = 1$ resulted only from the fact that, for simplicity of exposition, we had assumed proportional reinsurance. For more general reinsurance schemes, namely for more general functions $h(b, Y)$ one may obtain also optimal values that coincide neither with $b_{\min}$ nor with 1 and this is consistent with known results.*

R.2 *Based on arguments in Remark 1 in Diasparra and Romera (2010) we had restricted ourselves to stationary strategies. By relaxing this restriction, one would then, instead of maximizing $E\{V_t^\pi\}$, choose adaptively at each event time T_n the strategy $\pi_n = (b_n, \delta_n)$ to maximize*

$$E\{V_n^\pi \mid K_{T_n}\} = C(b_n)E\{Z_1 \mathbf{1}_{\{Z_1 < T - T_n\}}\}$$
$$-(1 - E\{K_{T_{n+1}} \mid K_{T_n}\})b_n E\{Y_1\} + E\{K_{T_{n+1}} \mid K_{T_n}\}\delta_n(E\{e^{W_1}\} - 1) \tag{29}$$

which results from the fact that (Z_n, Y_n, W_n) were supposed i.i.d. independent of the strategy π and the transition probability matrix $P = \|p_{ij}\|_{i,j\in\{0,1\}}$ is independent of the strategy as well. This allows us in fact to maximize $E\{V_t^\pi\}$ with an adapted strategy π (the only adaptation is to the currently observed values of K_{T_n}) by maximizing with a forward procedure the right hand side in (29). The remaining implications are the same as those implied above by the stationary optimal strategy except that the bound might be improved given that the stationary strategies are a subclass of the adapted strategies. Given that the two optimization problems are very similar (differ only by the adaptation to K_{T_n}), the bounds will not differ by too much, especially under the stationarity Assumption 11.

5 Exponential claims

The crucial step to obtain the bounds in Theorem 10 is to obtain a solution to equation (15). We investigate now this equation in the special, but relevant case when Z, Y and W are independent random variables, whereby Z is distributed negative-exponential with parameter $1/\lambda > 0$ and Y negative-exponential with parameter $1/\eta > 0$, i.e.

$$\begin{aligned} Z &\sim f_\lambda(z) = \frac{1}{\lambda}e^{-\frac{z}{\lambda}}, z \geq 0, \\ Y &\sim f_\eta(y) = \frac{1}{\eta}e^{-\frac{y}{\eta}}, y \geq 0 \end{aligned}$$

For W we only assume that its support, which (see the description after (3)) we consider given by the interval $[\underline{w}, \bar{w}]$, includes both positive and negative values (prices may go up or down); we also allow for its distribution to have a positive mass at zero implying that, even if we are at a jump time for the price, the price may actually not move (in this context see also Edoli and Runggaldier (2010)). Our main purpose here is to compare our bounds according to Theorem 10 with the classical Lundberg bound that, in fact, concerns the case of exponential claims.

5.1 Specific form for Equation (15)

We proceed along two steps: first assuming, as we do in the rest of the paper, a finite horizon $T > 0$; second, investigating the solution to (15) in the limit when $T \uparrow \infty$. This solution will then be a basis for the comparison with the Lundberg bound in the next subsection 5.2.

5.1.1 Equation (15) for finite horizon $T > 0$

We have the following

Proposition 15 *Assume that Z and Y are exponentially distributed with parameters $\frac{1}{\lambda}$ and $\frac{1}{\eta}$ respectively and $W \in [\underline{w}, \bar{w}]$ a given random variable. For a fixed $t \in [0,T]$, $\pi = (b, \delta)$, a given $k_0 = K_{T_{N_t}}$ and for $0 < r < \bar{r} := \frac{1}{b\eta}$ the function $\ell^{\pi}_{k_0}(r)$ in (14) admits the representation*

$$\begin{aligned} \ell^{\pi}_{k_0}(r) &= \left(\frac{1}{\lambda r C(b) + 1} \left[1 - e^{-(T-t)(rC(b) + \frac{1}{\lambda})} \right] - e^{-\frac{T-t}{\lambda}} \right) \\ &\quad \cdot \left(p_{k_0 0} \frac{1}{1 - rb\eta} + p_{k_0 1} \right) \\ &\quad \cdot \left(p_{k_0 0} + p_{k_0 1} \, e^{r\delta} M_{e^W}(-r\delta) \right) - 1 \end{aligned} \tag{30}$$

The proof of the Proposition is given in the Appendix. Notice also that (see the proof of the Proposition) the factor $\frac{1}{1-rb\eta}$ in (30) results from the moment generating function $E\left\{e^{rbY_1}\right\}$ and this is the reason for the restriction $0 < r < \bar{r} := \frac{1}{b\eta}$.

As can be seen from (30), even for the given standard distributions, equation (15) results in a rather complicated expression, for which it is difficult to obtain an explicit solution. In the absence of an explicit expression for R^{π}_0 to be used to obtain the bound in Theorem 10, it appears reasonable to compare, as we shall do in the next subsection 5.2, the bound obtained in this paper with the classical Lundberg bound in the exponential case and for large values of T, i.e. in the limit when $T \uparrow \infty$.

5.1.2 Asymptotic expression for equation (15) when $T \uparrow \infty$

A more convenient asymptotic expression for equation (15) can be obtained on the basis of Assumption 11. Under this assumption there is in fact no more statistical dependence of K_{T_n} on $K_0 = k_0$ and we have $\mathbf{1}_{\{Z_1 < T-t\}} = 1$ a.s.

From (30) in Proposition 15 we immediately obtain

Corollary 16 *For Z and Y exponentially distributed with parameters $\frac{1}{\lambda}$ and $\frac{1}{\eta}$ respectively and $W \in [\underline{w}, \bar{w}]$ a given random variable, under Assumption 11 we have*

$$\lim_{T\to\infty} (\ell^\pi_{k_0}(r)+1) = \frac{1}{1+rC(b)\lambda}[Q_0 \frac{1}{1-rb\eta} + Q_1][Q_0 + Q_1.e^{r\delta} M_{e^W}(-r\delta)]. \quad (31)$$

uniformly in $r \in (0, \frac{1}{b\eta})$.

In the next section 5.2 we discuss a sufficient condition for which, in the exponential case and asymptotically for $T \uparrow \infty$, our bound according to Theorem 10 is stronger than the Lundberg bound. Since, according to Assumption 11 and the ensuing Corollary 16, the function $\ell^\pi_{k_0}(r)$ in (14) does not anymore depend on k_0, we rewrite equation (15) in the form

$$\ell^\pi(R^\pi) = 0 \quad (32)$$

and consider its solution R^π as the value to be used in the asymptotic bound in Theorem 10.

5.2 Comparison with the Lundberg bound (exponential case and asymptotically for $T \uparrow \infty$)

Assume in the standard risk model that Y is a negative-exponential random variable with parameter $1/\eta$ and moment-generating function $M_Y(r)$. Given an initial wealth x, the classical Lundberg bound on the ruin probability is given by $e^{-R_L x}$, where R_L is the solution of

$$M_Y(r) - 1 = \frac{c}{1/\eta} r$$

namely of

$$\frac{1}{1-r\eta}-1=\frac{cr}{1/\eta}, \quad r<1/\eta$$

where c is the premium rate which, according to iii) in Assumption 1 (see also point ii) in the ensuing Remark 2), satisfies $c \geq 1$.

The value R_L with

$$\frac{1}{\eta} > R_L = (1/\eta)(1-1/c) > 0 \tag{33}$$

is thus the Lundberg *adjustment coefficient* that refers to the classical risk model without reinsurance and investment in the financial market (see e.g. Asmussen (2000)). This bound is evaluated for the case when the insurer pays the total amount of the claim, i.e. $b=1$, but in fact $b \in [b_{\min}, 1]$ and the so the actual amount paid by the insurer comes from a negative-exponential distribution with parameter $\frac{1}{b\eta}$. We thus have to consider this fact in the *Lundberg adjustment coefficient* so that it can be considered to satisfy, instead of (33), the following

$$\frac{1}{b\eta} > R_L = (1/b\eta)(1-1/c) > 0. \tag{34}$$

Our purpose is now to compare, in the exponential case and for $T \to \infty$, the Lundberg bound $e^{R_L x}$ with the bound $e^{R^\pi x}$ that we obtain according to Theorem 10. We shall in fact discuss sufficient conditions for which our bound is sharper, namely $R_L < R^\pi$. These conditions involve also the stationary strategy $\pi = (b, \delta)$, in particular the investment δ in the financial market, .

To this effect recall from Proposition 6 that the function $\ell^\pi_{k_0}(\cdot)$ is convex with a negative slope at the origin and it has a unique positive root in R^π_0. The same holds evidently true also for $\ell^\pi(\cdot)$ in (32) and its root R^π. Checking for $R_L < R^\pi$ becomes then equivalent to verifying whether $\ell^\pi(R_L) < 0$. We shall actually verify this latter condition in the equivalent form

$$\ell^\pi(R_L) + 1 < 1 \tag{35}$$

where, we recall, $0 < \frac{1}{b\eta}\left(1-\frac{1}{c}\right) = R_L < \frac{1}{b\eta}$.

From (31) in Corollary 16 we now have that, asymptotically for $T \to \infty$,

$$\begin{aligned} &\ell^\pi(R_L)+1)+1 \\ &= \left[\tfrac{1}{1+R_L C(b)\lambda}\right][Q_0 \tfrac{1}{1-R_L b\eta} + Q_1][Q_0 + Q_1.e^{R_L \delta} M_{e^W}(-R_L \delta)]. \end{aligned} \tag{36}$$

5.2.1 No investment in the financial market

We first consider the case of no investment in the financial market, i.e. $\delta = 0$, for which the third factor on the right hand side in (36) is equal to 1 so that condition (35) reduces to verifying whether the product of the first two factors in (36) is less than 1. We have

Proposition 17 *Assuming $\delta = 0$, the following holds*

i) *If $\frac{\lambda\,(c-1)}{\eta\,c} < 1$ then one can choose c^* such that $b_{\min} \leq \frac{\lambda\,(c-1)}{\eta\,c}$. For all reinsurance levels for which*

$$b_{\min} \leq b \leq \frac{\lambda\,(c-1)}{\eta\,c} \tag{37}$$

one then has that property (35) holds and, furthermore, these values of b satisfy $b\eta - \lambda < 0$.

ii) *If $b_{\min}\eta - \lambda > 0$ then condition (35) cannot be assured as long as $\delta = 0$.*

Proof: For $\delta = 0$ and recalling that $R_L b\eta < 1, \lambda > 0, Q_0 + Q_1 = 1, c^* \leq C(b) \leq c$, equation (36) becomes

$$\begin{aligned}
\ell^\pi(R_L) + 1 &= \frac{1}{1 + R_L C(b)\lambda}(Q_0 \frac{1}{1 - R_L b\eta} + Q_1) \\
&= \frac{Q_0 + Q_1(1 - R_L b\eta)}{(1 + R_L C(b)\lambda)(1 - R_L b\eta)} \\
&= \frac{1 - Q_1 + Q_1(1 - \frac{1}{b\eta}(1 - \frac{1}{c})b\eta)}{(1 + \frac{1}{b\eta}(1 - \frac{1}{c})C(b)\lambda)(1 - \frac{1}{b\eta}(1 - \frac{1}{c})b\eta)} \\
&= \frac{1 - Q_1(1 - \frac{1}{c})}{(1 + \frac{1}{b\eta}(1 - \frac{1}{c})C(b)\lambda)(1 - (1 - \frac{1}{c}))} \\
&= \frac{1 - Q_1(1 - \frac{1}{c})}{(1 + \frac{1}{b\eta}(1 - \frac{1}{c})C(b)\lambda)\frac{1}{c}} \leq \frac{c(1 - Q_1(1 - \frac{1}{c}))}{(1 + \frac{1}{b\eta}(1 - \frac{1}{c^*})c^*\lambda)} \\
&= \frac{c(1 - Q_1) + Q_1}{(1 + \frac{1}{b\eta}(1 - \frac{1}{c^*})c^*\lambda)} \leq \frac{c + Q_1}{1 + \frac{\lambda}{b\eta}(c^* - 1)}
\end{aligned} \tag{38}$$

To obtain property (35) it thus suffices that

$$c \leq Q_0 + \frac{\lambda}{b\eta}(c^* - 1) \tag{39}$$

This condition (39) has to be made compatible with $0 < c^* \leq c$ where $c > 1$. For this it suffices that c, b, η, λ are such that

$$c \leq Q_0 + \frac{\lambda}{b\eta}(c^* - 1) \leq Q_0 + \frac{\lambda}{b\eta}(c - 1) \tag{40}$$

A sufficient condition for (40) to hold, namely that there exists c^* with $0 < c^* \leq c$ satisfying that condition is

$$c\,(b\eta - \lambda) < Q_0 b\eta - \lambda \tag{41}$$

Notice that condition (40) implies $\frac{\lambda}{b\eta} > 1$ or $b\eta - \lambda < 0$ (thereby showing the last statement in point i) of the Proposition) and so from (41) we also obtain, as it should $c > \frac{\lambda - b\eta}{\lambda - Q_0 b\eta} \geq 1$.

Under the condition in point i) of the Proposition we now see that if we choose b according to (37) (recall that $C(b)$ is monotonically increasing in b with $C(b_{\min}) = c^* > 0$ and $C(1) = c > 1$), then $c\,(b\eta - \lambda) \leq \lambda(c-1) - c\lambda = -\lambda$ and so, a fortiori, we have (41) and with it the required property (35).

On the other hand, under the condition ii) of the Proposition we have $b\eta - \lambda > 0$ for all $b \in [b_{\min}, 1]$. In this case, since $c > 1$ and $Q_0 < 1$, the sufficient condition (41) does not hold and we are unable to state whether condition (35) holds unless we involve also the third factor in equation (36), i.e. we invest in the financial market. ∎

5.2.2 Investment in the financial market

Assume now that the product of the first two terms on the right side in (36) cannot be guaranteed to be less than 1 and let us assume that it is equal to a value $\frac{1}{\Gamma} > 1$. We now show that, if the financial market situation is favorable then by investing appropriately according to a $\delta > 0$, we can make the third factor on the right in (36) less than $\Gamma < 1$ so that (35) holds. To simplify the technical aspects we assume that W is a discrete random variable taking the three values $W \in \{\underline{w}, 0, \bar{w}\}$ with probabilities p_1, p_2, p_3 respectively (more generally we could have a continuous random variable with a point mass of p_2 at 0 and with $P\{W \in [\underline{w}, 0)\} = p_1$, $P\{W \in (0, \bar{w}] = p_3)$. We have

Proposition 18 *Assume that the market situation is favorable in the sense that for the probability of a price increase we have*

$$p_3 \geq \frac{\exp\left[-\frac{1}{\eta}\left(1-\frac{1}{c}\right)\delta\left(e^{\underline{w}}-1\right)\right]-\Gamma}{\exp\left[-\frac{1}{\eta}\left(1-\frac{1}{c}\right)\delta\left(e^{\underline{w}}-1\right)\right]-\exp\left[-\frac{1}{\eta}\left(1-\frac{1}{c}\right)\delta\left(e^{\bar{w}}-1\right)\right]} \tag{42}$$

then by choosing the amount δ to be invested in the financial market such that

$$\delta > -\frac{\log\Gamma}{\frac{1}{\eta}\left(1-\frac{1}{c}\right)\left(e^{\bar{w}}-1\right)} \tag{43}$$

which makes the right hand side of (42) less than 1, we have that the third factor in (36) is less than Γ (we assume implicitly that the upper bound $\bar{\delta}$ is chosen to be larger than the expression on the right side in (43).)

Proof: Written in explicit form, the requirement for the third factor in (36) to be less than Γ becomes

$$\begin{aligned} & e^{R_L\delta}M_{e^W}(-R_L\delta) = \\ & = p_1 e^{-R_L\delta(e^{\underline{w}}-1)} + p_2 + p_3 e^{-R_L\delta(e^{\bar{w}}-1)} \\ & = (1-p_2-p_3)\,e^{-R_L\delta(e^{\underline{w}}-1)} + p_2 + p_3 e^{-R_L\delta(e^{\bar{w}}-1)} \leq \Gamma < 1 \end{aligned} \tag{44}$$

namely that

$$\begin{aligned} & p_3\left[e^{-R_L\delta(e^{\bar{w}}-1)} - e^{-R_L\delta(e^{\underline{w}}-1)}\right] + p_2\left[1-e^{-R_L\delta(e^{\underline{w}}-1)}\right] \\ & \quad \leq \Gamma - e^{-R_L\delta(e^{\underline{w}}-1)} \end{aligned} \tag{45}$$

a condition that, as can easily be seen, cannot be satisfied for $\delta = 0$. It is also easily seen that, for (45) to be satisfied, it suffices that

$$p_3\left[e^{-R_L\delta(e^{\underline{w}}-1)} - e^{-R_L\delta(e^{\bar{w}}-1)}\right] \geq e^{-R_L\delta(e^{\underline{w}}-1)} - \Gamma \tag{46}$$

and for this we have to require that

$$p_3 \geq \frac{e^{-R_L\delta(e^{\underline{w}}-1)} - \Gamma}{e^{-R_L\delta(e^{\underline{w}}-1)} - e^{-R_L\delta(e^{\bar{w}}-1)}} \tag{47}$$

which, recalling that $R_L = \frac{1}{\eta}\left(1-\frac{1}{c}\right)$, becomes (42). It remains to make sure that the right hand side in (47) is less than 1, which can easily be seen to be the case if $e^{-R_L\delta(e^{\bar{w}}-1)} < \Gamma$, a condition that reduces to (43) when replacing R_L with its explicit expression. ∎

6 Appendix: *Proof of Proposition 15*

By (13) we have, for $t \in [0, T]$ given, that (14) is equivalent to

$$\begin{aligned}
\ell^{\pi}_{k_0}(r) + 1 &= E\{e^{-rV^{\pi}_t} | k_0\} \\
&= E\{\exp(-rC(b)Z_1 \mathbf{1}_{\{Z_1 < T-t\}}) | k_0\}.E\{\exp(r(1 - K_{T_{N_t+1}})bY_1) | k_0\} \\
&\quad \cdot E\{\exp(-rK_{T_{N_t+1}}\delta(e^{W_1} - 1)) | k_0\} \\
&= I \cdot II \cdot III
\end{aligned} \tag{48}$$

We now consider I, II and III separately.

- Factor I : Note that the random variable $\{Z_1 \mathbf{1}_{\{Z_1 < T-t\}}\}$ is independent of the event $K_0 = k_0$.

$$\begin{aligned}
I &= E\{\exp(-rC(b)Z_1 \mathbf{1}_{\{Z_1 < T-t\}}) | k_0\} = \int_0^{\infty} e^{-rC(b)z\mathbf{1}_{\{z<T-t\}}} \frac{1}{\lambda} e^{-\frac{z}{\lambda}} dz \\
&= \int_0^{T-t} e^{-rC(b)z} \frac{1}{\lambda} e^{-\frac{z}{\lambda}} dz + \int_{T-t}^{\infty} \frac{1}{\lambda} e^{-\frac{z}{\lambda}} dz \\
&= \frac{1}{\lambda} \int_0^{T-t} e^{-z(rC(b)+\frac{1}{\lambda})} dz + \int_{\frac{T-t}{\lambda}}^{\infty} e^{-w} dw \\
&= \frac{1}{\lambda(rC(b) + \frac{1}{\lambda})} \int_0^{(T-t)(rC(b)+\frac{1}{\lambda})} e^{-s} ds + \int_{\frac{T-t}{\lambda}}^{\infty} e^{-w} dw \\
&= \frac{1}{\lambda rC(b) + 1} \left[1 - e^{-(T-t)(rC(b)+\frac{1}{\lambda})}\right] - e^{-\frac{T-t}{\lambda}}.
\end{aligned} \tag{49}$$

- Factor II : Note that in this case, we do not have independence of the random variable $\{(1 - K_{T_{N_t+1}})bY_1\}$ from the event $K_0 = k_0$.We make use of the moment-generating function of a negative-exponential random variable with parameter $\frac{1}{\eta}$, i.e

$$M_Y(s) = E\{e^{sY}\} = (1 - s\eta)^{-1}, \text{ for } s < 1/\eta.$$

Thus, we have

$$\begin{aligned}
II &= E\{e^{r(1-K_{T_{N_t+1}})bY_1} | k_0\} = E\{E\{e^{(1-K_{T_{N_t+1}})rbY_1} | K_{T_{N_t+1}}, k_0\} | k_0\} = \\
&= \Pr\{K_{T_{N_t+1}} = 0/k_0\} E\{e^{rbY_1}\} + \Pr\{K_{T_{N_t+1}} = 1/k_0\} = \\
&= p_{k_0 0} \frac{1}{1 - rb\eta} + p_{k_0 1}, \quad \text{being } 0 < r < 1/b\eta.
\end{aligned} \tag{50}$$

- Factor III : Note that in this case, we do not have independence of the random variable $\{K_{T_{N_t+1}}\delta(e^{W_1}-1)\}$ from the event $K_0 = k_0$.

$$\begin{aligned} III &= E\{\exp(-rK_{T_{N_t+1}}\delta(e^{W_1}-1))|k_0\} = E\{E\{e^{(-rK_{T_{N_t+1}}\delta(e^{W_1}-1))}|K_{T_{N_t+1}}, k_0\}|k_0\} \\ &= \Pr\{K_{T_{N_t+1}} = 0/k_0\} + \Pr\{K_{T_{N_t+1}} = 1/k_0\}E\{e^{-r\delta(e^{W_1}-1)}\} \\ &= p_{k_0 0} + p_{k_0 1}\, e^{r\delta} M_{e^W}(-r\delta), \end{aligned} \tag{51}$$

The statement now follows. ∎

References

[1] S. Asmussen (2000), Ruin Probabilities, World Scientific, River Edge, NJ.

[2] M.A. Diasparra and R. Romera, Bounds for the ruin probability of a discrete-time risk process, J. Appl. Probab.(2009) Vol. 46, 1 , 99-112.

[3] M. Diasparra and R. Romera, Inequalities for the ruin probability in a controlled discrete-time risk process, European Journal of Operational Research (2010), Vol. 204, 3, 496-504.

[4] M. Schäl, On Discrete-Time Dynamic Programming in Insurance: Exponential Utility and Minimizing the Ruin Probability, Scandinavian Actuarial Journal, (2004), Vol. 3,189 – 210.

[5] M. Schäl, Control of ruin probabilities by discrete-time investments, Math. Meth. Oper. Res. (2005), Vol. 62, 2, 141-158.

[6] H. Schmidli, On minimizing the ruin probability by investment and reinsurance, Ann. Appl. Probab., (2002), Volume 12, 890-907.

[7] H. Schmidli (2008), Stochastic Control in Insurance. Springer, London.

[8] G. Willmot and X. Lin (2001), Lundberg Approximations for Compound Distributions with Insurance Applications (Lectures Notes Statist. 156), Springer, New York.

[9] E. Edoli and W.J. Runggaldier, On Optimal Investment in a Reinsurance Context with a Point Process Market Model, Preprint (2010). To appear in Insurance: Mathematics and Economics.

[10] J.Grandell (1991) Aspects of Risk Theory. Springer, New York.

[11] S. Chen, H. Gerber and E. Shiu , Discounted probabilities of ruin in the compound binomial model, Insurance: Mathematics and Economics (2000), Vol. 26, 239-250.

[12] R. Wang, H. Yang and H. Wang, On the distribution of surplus inmediately after ruin under interest force and subexponential claims (2004), Insurance: Mathematics and Economics 35, 703-714.

[13] T. Huang, R. Zhao and W. Tang , Risk model with fuzzy random individual claim amount, European Journal of Operational Research (2009), 192, 879-890.